Does Nash Envy Immunity?  
Ching-Hua Yu  
Department of Computer Science, University of Illinois, Urbana-Champaign

The Story

- Envy is ascribed to the humanity nature of comparison. (Partly rational)
- The (un)happiness of employees can depend on, instead of their own salary, the comparison between their salary and their colleagues.
- A (somewhat) irrational bidder in an auction can set a bid higher than the true value of an item just because he is unhappy to see his rival bidder win the item.
- In the “Millionaires’ Problem,” the millionaires care which of them is richer.

In contrast, Byzantine players target on impairing the other’s happiness. (Completely irrational)

Basic Result

- Nash Equilibrium, Envy-Proofness, and Immunity
  - Theorem (Immune + Nash = Envy-proof). Every immune Nash equilibrium in $G = ([x_1, x_2], \{A_1, A_2\})$ is envy-proof.
  
- Envy occurs when $x_1 < x_2$, but $x_1^{A_1} > x_2^{A_2}$.
  
- Robust existence.
  - Every finite two-player game admits an immune profile.
  - Every finite two-player game admits an envy-proof profile.
  - There are two-player games which do not admit any envy-proof Nash equilibrium.

Solving Immunity + Nash Equilibrium is Easy

- Theorem. Let $G = ([x_1, x_2], \{A_1, A_2\})$ be a normal-form 2-player game. It is polynomial-time solvable to determine whether there exists an immune Nash equilibrium in $G$ and to output a solution if there exists.

Solution Idea: $x$ is an immune Nash equilibrium if and only if $\forall y \in [x_1, x_2], \forall i = 1,2, x_i^{A_i} < x_i(x)$.

(N)Non-Existence

- There are two-player games which do not admit any immune Nash equilibrium.

Figure 1. A coordination game (movie, shopping) is the only Nash equilibrium which is envy-proof, whereas shopping is the only envy-proof-strategy profile.

Is Finding Envy-proof Profiles or Immune Profiles easier than Finding Nash Equilibrium?

1. Finding an immune profile in a finite two-player game is PPAD-complete.  
2. Finding an envy-proof profile in a finite two-player game is polynomial-time solvable.

Solving Envy-Proofness + Nash Equilibrium is hard

- Theorem. Even in symmetric 2-player games, it is $\text{NP}$-complete to determine whether there is an envy-proof Nash equilibrium.

Proof Idea - Reduction from SATISFIABILITY. Construct a game such that every legal assignment of a Boolean formula corresponds to an envy-proof profile in the game, and every Nash equilibrium, except a default Nash equilibrium, corresponds to a satisfiable assignment of the formula. Meanwhile, the default, always-existing Nash equilibrium serves as an absorbing state in the game but is not envy-proof.

Approximation

- Definition (Approximation). Let $G = ([x_1, x_2], \{A_1, A_2\})$ be a 2-player game and $x$ be a strategy profile.

  - $x$ is an $\epsilon$-Nash-equilibrium if $\forall y \in [x_1, x_2] \setminus x, x_i^{A_i} < x_i(x) + \epsilon$.
  - $x$ is immune if $\forall y \in [x_1, x_2], x_i^{A_i} \geq y_i(x) - \epsilon$.

- E.g., an immune profile says that each player's unilateral deviation cannot decrease the rival player's utility more than $\epsilon$ (in absolute value). Intuitively, an approximate immune profile is stable against a "lazy" Byzantine player who attacks only when he can make sufficiently huge loss of his opponent.

Pseudo-Polynomial Solution

Lemma (Approximate Immune Profile). For a 2-player game $G = ([x_1, x_2], \{A_1, A_2\})$, there is a $O^{\text{poly}(N)}$ algorithm to find an $\epsilon$-immune profile.

Theorem (Approximate Envy-Proof NE). Let $G = ([x_1, x_2], \{A_1, A_2\})$ be a 2-player game with $|N| = n$, $x = (\gamma_1, \gamma_2)$, where $\gamma_i$ is the minimum approximate Nash equilibrium of $G$ and $\gamma_n = \gamma_n - \frac{|A_1| - |A_2|}{|A_1|}$.

1. There is a $O^{\text{poly}(N)}$ time algorithm for computing a $\epsilon$-envy-proof $\epsilon$-Nash equilibrium, where $\epsilon$ is the minimum approximation factor of envy-proof profiles over all Nash equilibrium.

2. If $N > 2$, then there is no $O^{\text{poly}(N)}$ envy-proof Nash equilibrium in $G$.

Multiplayer Games

Games of Special Interest

- Anonymous games
- $y$-sensitive games [GR08, KPRUl13]

Definition. A game $G = ([x_1, x_2], \{A_1, A_2\})$ is said to be $y$-sensitive if $\forall y \in [x_1, x_2]: x_i^{A_i} \geq x_i(y) - \epsilon(y)$.

- $y$-varied game

Definition. A game $G = ([x_1, x_2], \{A_1, A_2\})$ is said to be $y$-varied if $\forall y \in [x_1, x_2]: x_i^{A_i} \geq x_i(y) - 1$.

Coalescent Envy-Proofness

Definition (p-t-coalitional-envy-proofness). In a player game $G = ([x_1, x_2], \{A_1, A_2\})$, a strategy profile $x = (x_1, x_2) \leq (x_1, x_2)$, if (coalitional envy-proof) for every set $S$ of the agents $1, \forall x_i \in S, \forall y \in [x_1, x_2]$, $x_i^{A_i} \geq x_i(y)$ for $i \neq S$.

Results

1. Let $G$ be an $n$-player $n$-game. Then every Nash equilibrium of $G$ is an envy-proof Nash equilibrium.

2. Let $G$ be an $n$-player game. Then every Nash equilibrium of $G$ is a $y$-sensitive Nash equilibrium.

3. For every natural number $n$, every $c \geq 1$, there exists a constant such that for every 2-player game $G = ([x_1, x_2], \{A_1, A_2\})$ with $|A_1| = c$, $x_i^{A_i} \geq x_i(y) - c$ for all $y \in [x_1, x_2]$. $G$ admits a $c$-Nash equilibrium which is in $O(c)$-coalitional envy-proof.

Discussion and Future Work

- Special topics
  - Routine, auctions, markets, etc. are natural applications/extensions to discuss these notions.
  - Multiplayer games
  - Many questions left for general and special multiplayer games
  - Beyond equilibria
  - Other factors such as Social welfare
  - Incomplete information in an adversarial environment
  - Without a coordinator, is there certain dynamics that approximate these stable solutions when there are?  
  - Rational Cryptography
  - Rational players are between rational players and Byzantine players.
  - Cross the impossibility or get better trade-off
  - Protocols of multiparty computation for enriched player components can have applications in games as well.